Chapter 6

Functions

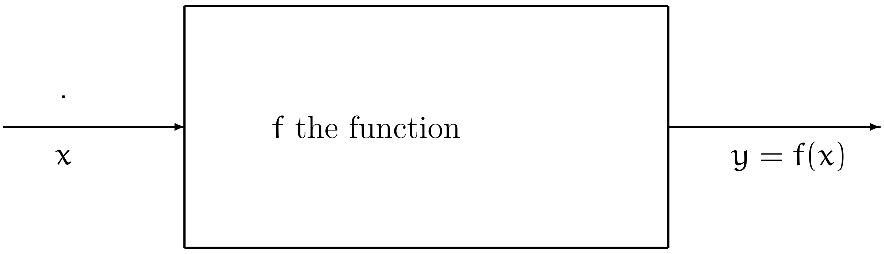
> *Mathematicians are like Frenchmen: whatever you say to them they translate into their own language and forthwith it is something entirely different.*
>
> Johann Wolfgang von Goethe

One of the most fundamental (and useful) ideas in mathematics is that of a *function.* As a preliminary definition suppose we have two sets X and $Y$and we also have a rule which assigns to every $x \in X$ a UNIQUE value $y \in Y$. We will call the rule f and say that for each x there is a $y = f(x)$ in the set Y. This is a very wide definition and one that is very similar to that of a relation , the critical point is that for each a there is a *unique value* y. A common way of writing functions is

$$f : X \to Y$$

which illustrates that we have two sets X and Y together with a rule f giving values in Y for values in X. We can think of the pairs (x, y) or more clearly $(x, f(x))$. This set of pairs is the *graph* of the function

In what follows we show how functions arise from the idea of relations and come up with some of the main definitions. You need to keep in mind the simple idea a function is a rule that takes in x values and produces y values. It is probably enough to visualize f as a device which when given an x value produces a y.


x
f the function
y = f(x)

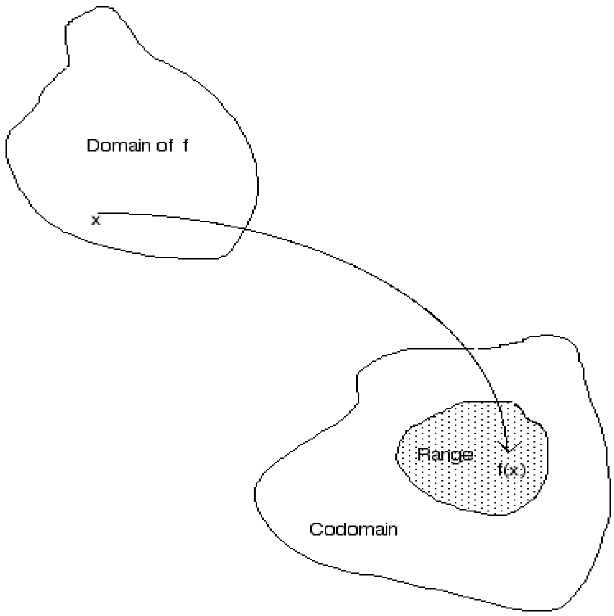


Figure 6.1: Function f

Clearly if you think of f as a machine we need to take care about what we are allowed to put in, x, and have a good idea of the range of what comes out, y. It is these technical issues we look at next.

The set X is called the domain of the function f and Y is *codomain*. We are normally more interested in the set of values $\{ f(x) : x \in X\}$. This is the *range* R sometimes called the *image* of the function. See figure 6.1

Examples

We can have

$$f : X \to Y$$

where

1. $f(x) = 2^x$ where $X = \{x : 0 \le x < \infty\}$ and $Y = \{y : 0 \le x < \infty\}$
2. $f(x) = \sqrt{x}$ where $X = \{x : 0 \le x < \infty\}$ and $Y = \{y : 0 \le y < \infty\}$
3. $f(x) = sin^{-1}(x)$ where $X = \{x : -\pi/2 \le x < pi/2\}$ and $Y = \{-1 \le y \le 1\}$

If we think of the possibilities we have

- There may be some points in Y (the codomain) which cannot be reached by function f. If we take all the points in X and apply f we get a set

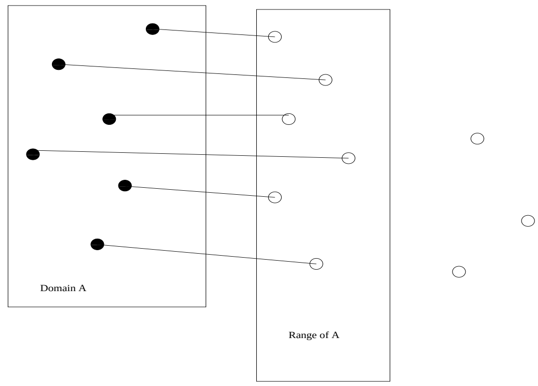


Figure 6.2: An onto function

$R = \{f(x) : x \in X\}$ which is the *range* of the function f. Notice R is a subset of Y i.e.$R \subset Y$.

- *Surjections* (or *onto* functions) have the property that for every y in the codomain there is an x in the domain such that $f(x) = y$. If you look at 6.1 you can see that in this case the codomain is bigger than the range of the function. See figure 6.2 If the range and codomain are the same then out function is a surjection. This means every y has a corresponding x for which $y = f(x)$
- Another important kind of function is the injection (or one-to-one function), which have the property that if $x_1 = x_2$ then y_1 must equal y_2. See figure 6.3
- Lastly we call functions *bijections*, when they are are both one-to-one and onto.

A more straightforward example is as follows. Suppose we define

$$f : X \to Y$$

where $f(x) = 2^x$ and $X = \{x : 0 \leq x < \infty\}$ and $Y = \{y : -\infty \leq x < \infty\}$. The range of the function is $R = \{y : 0 \leq x < \infty\}$ while the codomain Y has negative values which we cannot reach using our function.

Composition of functions

The *composition* of two or more functions uses the output of one function, say f, as the input of another, say g. The functions $f : X \to Y$ and $g : Y \to Z$ can be

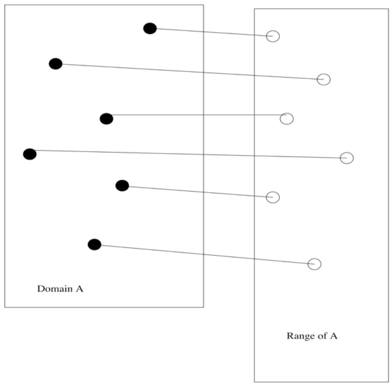


Figure 6.3: An 1 to 1 function

composed by applying f to an argument x to obtain $y = f(x)$ and then applying g to y to obtain $z = g(y)$. See figure 6.4. The composite function formed in this way from f and g can be written $g(f(x))$ or $g \circ f$. This last form can be a bit dangerous as the order can be different in different subjects. Using composition we can construct complex functions from simple ones, which is the point of the exercise.

One interesting function, given f, would be the function g for which x=g(f(x)). In other words g is the *inverse function*. Not all functions have inverses, in fact there is an inverse g written f^{-1} if and only if f is bijective. In this case $x = f^{-1}(f(x)) = f(f^{-1}(x))$.

The arrows and blob diagrams are not the usual way we draw functions. You will recall that the technical description of $f : X \to Y$ is the set of values $(x, f(x))$ Suppose we take the reals $\mathbb{R}$ so our function takes real values and gives us a new set of reals, say $f(x) = x^3$ we take x values , compute $y = f(x)$ for these values and plot them as in figure 6.6. Plotting functions is a vital skill, you know very little about a function until you have drawn the graph. It need not be very accurate, mathematicians often talk about *sketching* a function. By this they mean a drawing which is not completely accurate but which illustrates the main characteristics of the function,

Now we might reasonably does every sensible looking function have an inverse? An example consider $f(x) = x^2$ which is plotted in figure 6.8. There is now problem in the definition of f for all real values of x, that is the domain is $\mathbb{R}$ and the codomain $\mathbb{R}$. However if we examine the inverse we have a problem.

if we take y=4, this may arise from x=2 or x=-2. So there is not an $f^{-1} = y^{-1/2}$! If we change the domain we can get around this. Suppose we define $\mathbb{R}^+ = \{x :$

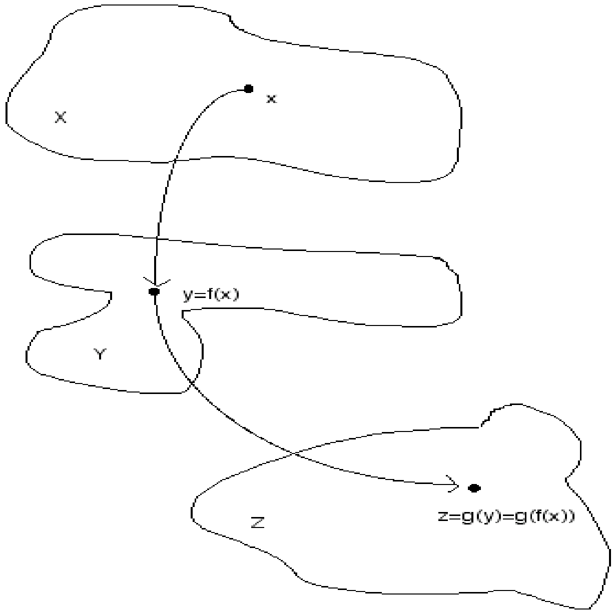


Figure 6.4: Composition of two functions f and g

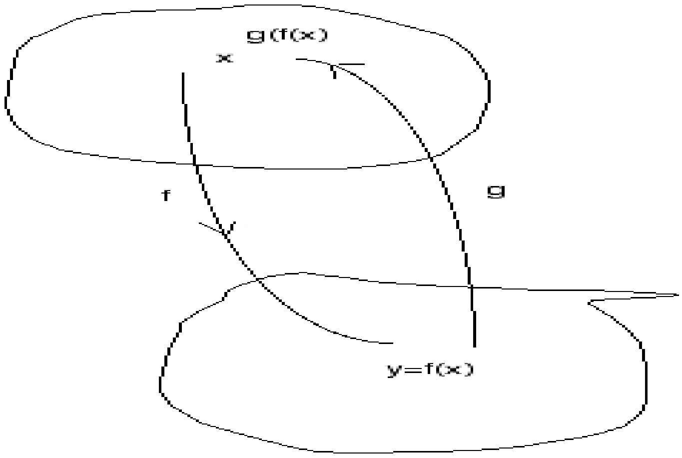


Figure 6.5: The inverse f and $g = f^{-1}$

Examples

1. Suppose $f(x) = x^2$ and $g(y) = 1/y$ then $g(f(x)) = 1/x^2$. We of course have to take care about the definition if the range and the domain to avoid $x = 0$

2. When $f(x) = x^2$ and $g(x) = x^{1/2}$ g is the inverse function when f is defined on the positive reals.

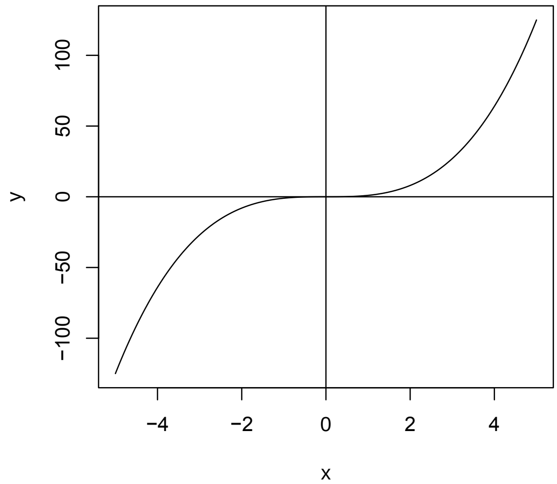


Figure 6.6: Plot of $f(x) = x^3$

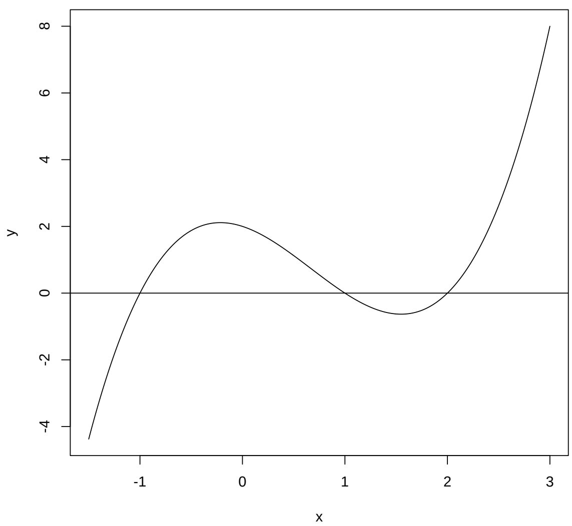


Figure 6.7: Plot of $f(x) = x^3 - 2x^2 - x + 2$

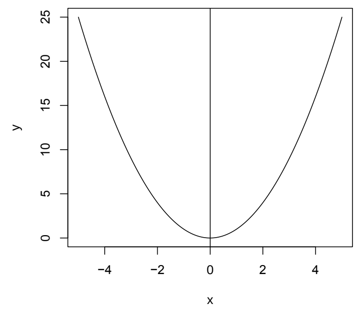


Figure 6.8: Plot of $f(x) = x^2$

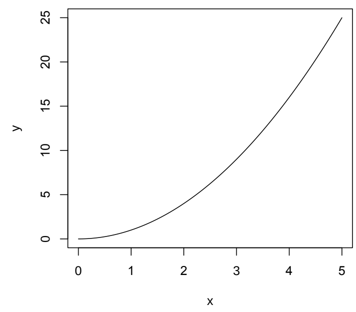


Figure 6.9: Plot of $f(x) = x^2$

$0 \leq x < \infty\}$ and consider $f(x) = x^2$ defined on $\mathbb{R}^+$ i.e.

$$\mathbb{R}^+ : f \to \mathbb{R}^+$$

In this case we do not have the problem of negative values of x. Every value of y arises from a unique x.

Exercises

For the following pairs evaluate $g(f(x))$ and $f(g(x))$.

1. $f(x) = 1/x, \; g(x) = x^2$
2. $f(x) = 3 + 4x, \; g(x) = 2x - 5$
3. $f(x) = x + 1, \; g(x) = x - 1$

6.0.8 Important functions

Over time we have come to see that some functions crop up again and again in applications. This seems a good point to look at some of these.

polynomials

We call functions like $f(x) = a_p x^p + a_{p-1} x^{p-1} + \ldots + a_1 x + a_0$ polynomials and these usually have a domain consisting of the reals. In out example the coefficients $a_0, a_1, \ldots, a_p$ are numbers and our polynomial is said to have order p. Examples are

1. $f(x) = x + 2$
2. $f(x) = x^3 - x^2 + x + 2$
3. $f(x) = x^{17} - 11$
4. $f(x) = x^2 - 3x + 2$

Zeros

Very often we need to know for what values of x for which $f(x) = a_p x^p + a_{p-1}x^{p-1} + \ldots + a_1 x + a_0 = 0$ is zero. The values are called the zeros or the roots of the polynomial. We can prove that a polynomial of degree p has at most p roots which helps a little. The simplest to way to find zeros is to factorize the polynomial so if

$$f(x) = x^3 - 6 * x^2 + 11x - 6 = (x-1)(x-2)(x-3)$$

so $f(x) = 0$ when $x = 1, 2, 3$.

Factorization is (as for integers) rather difficult. The best strategy is to try and guess one zero, say x=a and then divide the polynomial by (x-a). We then repeat. Polynomial division is just like long division. So to divide $x^3-6x^2+11x-6$ by $x-1$:

write out the sum

$$x-1\overline{\big)\ x^3-6x^2+11x-6}$$

find the power of x to multiply $x-1$

$$\begin{array}{rl} & x^2 \\ x-1\big) & \overline{x^3-6x^2+11x-6} \end{array}$$

multiply $x-1$ by x^2 as shown.

$$\begin{array}{rl} & x^2 \\ x-1\big) & \overline{x^3-6x^2+11x-6} \\ & \underline{-x^3 \ +x^2} \end{array}$$

subtract as shown.

$$\begin{array}{rl} & x^2 \\ x-1\big) & \overline{x^3-6x^2+11x-6} \\ & \underline{-x^3 \ +x^2} \\ & \quad -5x^2+11x \end{array}$$

find a multiplier to multiply $x-1$ to get a $-5x^2$

$$\begin{array}{rl} & x^2 \ -5x \\ x-1\big) & \overline{x^3-6x^2+11x-6} \\ & \underline{-x^3 \ +x^2} \\ & \quad -5x^2+11x \end{array}$$

multiply $x-1$ and subtract as shown

$$\begin{array}{rl} & x^2 \ -5x \\ x-1\big) & \overline{x^3-6x^2+11x-6} \\ & \underline{-x^3 \ +x^2} \\ & \quad -5x^2+11x \\ & \quad \underline{\ 5x^2 \ -5x} \\ & \qquad\qquad 6x-6 \end{array}$$

$$
\begin{array}{rr}
 & x^2 \quad -5x \\ \cline{2-2}
x-1) & x^3-6x^2+11x-6 \\
 & -x^3 \quad +x^2 \qquad\qquad \\ \cline{2-2}
 & -5x^2+11x \quad \\
 & 5x^2 \quad -5x \quad \\ \cline{2-2}
 & 6x-6
\end{array}
$$

find a multiplier to multiply $x-1$ to get a $6x$

$$
\begin{array}{rr}
 & x^2 \quad -5x+6 \\ \cline{2-2}
x-1) & x^3-6x^2+11x-6 \\
 & -x^3 \quad +x^2 \qquad\qquad \\ \cline{2-2}
 & -5x^2+11x \quad \\
 & 5x^2 \quad -5x \quad \\ \cline{2-2}
 & 6x-6 \\
 & -6x+6 \\ \cline{2-2}
 & 0
\end{array}
$$

nothing left so we stop!

The answer is x^2-5x+6. If there is something left then it is the remainder.

Hence

$$
\begin{array}{rr}
 & x-3 \\ \cline{2-2}
x-2) & x^2-5x+6 \\
 & -x^2+2x \quad \\ \cline{2-2}
 & -3x+6 \\
 & 3x-6 \\ \cline{2-2}
 & 0
\end{array}
$$

The answer is $x^2-5x+6=(x-2)(x-3)$.

However suppose we try

$$
\begin{array}{rr}
 & x-4 \\ \cline{2-2}
x-1) & x^2-5x+6 \\
 & -x^2 \quad +x \quad \\ \cline{2-2}
 & -4x+6 \\
 & 4x-4 \\ \cline{2-2}
 & 2
\end{array}
$$

We have a remainder and the answer is $x^2-5x+6=(x-1)(x-4)+2$.

Exercises

Factorize

1. $2x^3-x^2-7x+6$
2. $2x^3-3*x^2-5x+6$

The power function

Suppose we take values x from the reals and consider the function $P(x) = x^a$ for some value a. We can suppose that a is also real. So we have

$$\mathcal{R} : P \to \mathcal{R}$$

An example might be $P(x) = x^2$ or $P(x) = x^{1.5}$. In the second case we clearly have to redefine the domain. Can you see why? The properties of the power function

1. $x^a \times x^b = x^{a+b}$
2. $x^0 = 1$

Logarithms

We know that we can write powers of numbers, so

$$10^0 = 1 \quad 10^1 = 2 \qquad 10^2 = 100 \qquad 10^2 = 1000 \qquad \ldots$$

and $10^{0.5} = 3.162278\ldots$.

Now consider the backwards problem:

Given y can we find an x such that $y = 10^x$.

In other words if we define the power function $y = P(x) = 10^x$ for $x \in \mathcal{R}$, as above, then what is the inverse of this $P^{-1}(y)$? It may help to look at figure 6.10. We have plotted dotted lines from (1.5,0) to the curve. Going from x vertically to the curve and then to the y axis gives the power value $P(x) = y$. The reverse path from y to x is the logarithm.

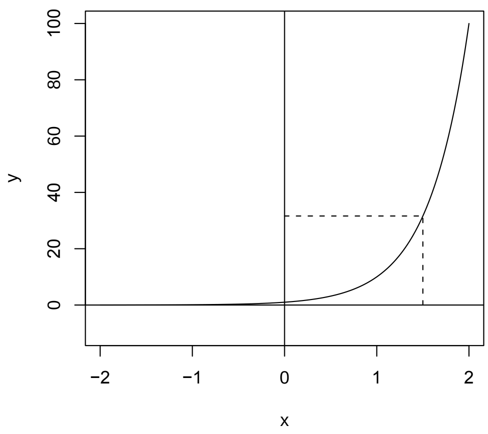


Figure 6.10: Plot of $f(x) = 10^x$

The inverse of $p(x)$ is call the logarithm or log and is written $\log_{10}(x)$. So

$$\log_{10}(1) = 0 \quad \log_{10}(10) = 1 \quad \log_{10}(100) = 0 \quad \log_{1000}(1) = \ldots$$

Often we are lazy and drop the 10 and just write $\log(x)$

Because we know that log is the inverse of the power function we have some useful rules

1. $\log(u) + \log(v) = \log(uv)$
2. $\log(u^v) = v \log(uv)$
3. $\log(u - \log(v) = \log\left(\frac{u}{v}\right)$
4. $-\log(u) = \log\left(\frac{1}{u}\right)$

Of course we did not have to choose 10 in our definitions. We could have choose 2, like many engineers, or any positive number a say. We then write $y = \log_a(x)$ to indicate the number y which satisfies $x = a^y$. The $\log_a(x)$ is called the log of x to *base* a.

For reasons which will (we hope) become apparent mathematicians like to use *natural logs* which have a base $e = 2.718282\ldots$. because they are used so often rather than write $\log_e(x)$ you will often see them written as $\ln(x)$ or just as $\log(x)$. All logs satisfy the rules set out in the list 6.0.8. We shall be lazy and just use logarithms to base e.

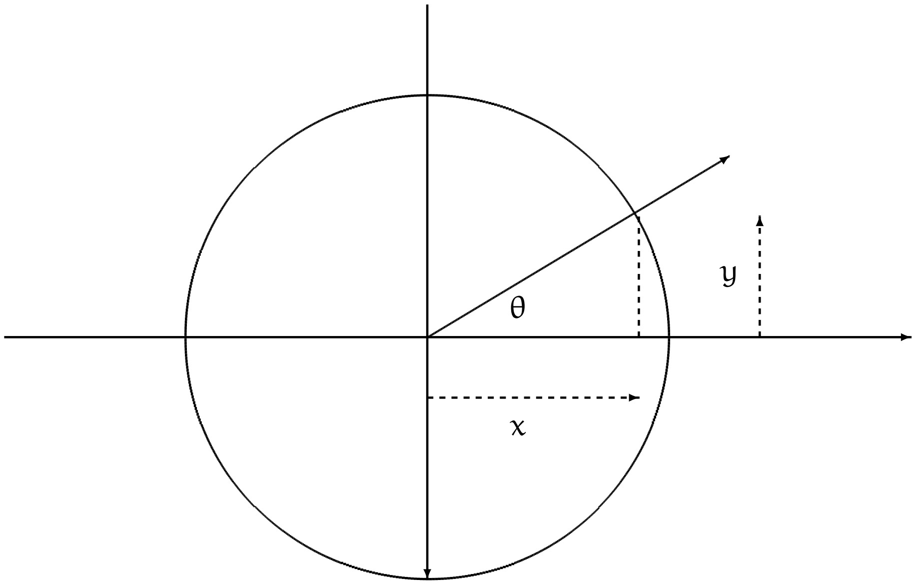
We can of course express logs in one base as logs in another. Suppose $x = a^{\log_a(x)} = b^{\log_b(x)}$ then taking logs gives

$$\log_a(x) = \log_a(b)\log_b(x)$$

Sometime it is natural to express powers as base 2 for example $y = P(x) = 2^x$. Mathematicians often use the number e so the power definition is $y = e^x$ which you will often see written as $y = \exp(x)$ since e^x is called the *exponential function.*

6.1 Functions and angular measure

We look briefly at the measurement of angles. Angular measure has been important from the very beginning of human history both in astronomy and navigation. Consider a circle with the angle θ made with the x axis as shown. Unlike maps in mathematics the reference line is *not* North but along the x axis and if we rotate anti-clockwise we sweep out an angle θ. The angle is traditionally measured in degrees, minutes and seconds. We will stick to degrees for the moment.



If we sweep anti-clockwise through 360 degrees we sweep out a circle. 180 degrees is a half circle and $720 = 3 \times 360$ two circles. Rotations in a clockwise direction are assumed to be negative degrees, so $-90^o = 270^o$

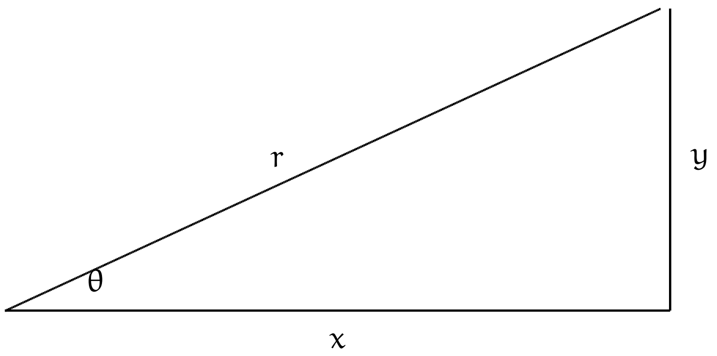
To complicate things a little we can also measure the angle in an equivalent way by measuring the length of the arc we make out on the circle as we sweep through the angle θ. Suppose this is s. For a circle of radius 1 s is a measure of the angle, although in different units called *radians.* So one circle is 2π radians and 90^o is $\pi/2$ radians. We convert from degrees to radians as follows

degrees	radians
θ	$2\pi\theta/360$
$360s/(2\pi)$	s

If you look at most "scientific calculators" you will see a button for switching from degrees to radians and vice versa.

The trigonometric functions

Of course we can measure angles in other ways. Suppose we look at the angle θ in the diagram. The ratio of the y and x values is related to the angle. Roman surveyors would often choose and angle by fixing the x value and the y value. As you can. imagine, five steps and then 3 steps vertically gives the same angle no matter where you are



Thus from the diagram θ is related to y/x. In fact we define y/x to be the tangent of θ written as $\tan\theta = y/x$. The inverse function is $\tan^{-1}\theta = y/x$ or sometimes $\arctan\theta = y/x$ The reader might like to examine our triable and see why the tangent of 90^o does not exist. We provide a plot of the tangent from 0 to just under 90 degrees in figure 6.11. If we keep the definition on the domain $0 \leq \theta < 90$ as is (relatively) simple. While the domain is easily extended we leave this to those of you will interests in this direction.

Of course we do not have to use tangents, although they are probably the most practical in applications. Alternative are to use the ratio y/r the height y divided by the radius of the circle r. This is called the sine function and written $\sin\theta = y/x$.

In a similar we we could use the cosine written $\cos\theta = x/r$. Both of these functions are plotted in figure 6.12 There are lots of links between these functions,

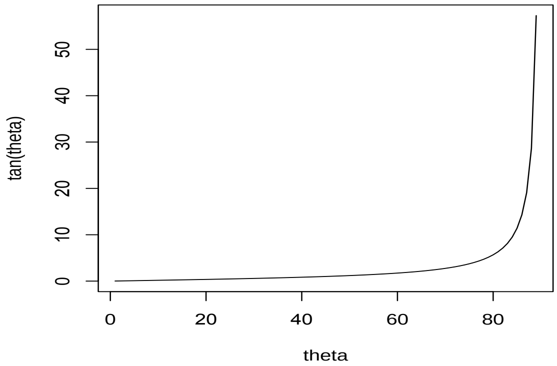


Figure 6.11: $\tan x$

for example

$$\tan\theta = \frac{\sin\theta}{\cos\theta}$$

This can be deduced quite simple from the definitions. Try it yourself!

The trigonometric functions are periodic in that if we plot them over a large part of the axis they repeat as in figure 6.13

Out next step is the study of the shapes of functions which brings us to Calculus.

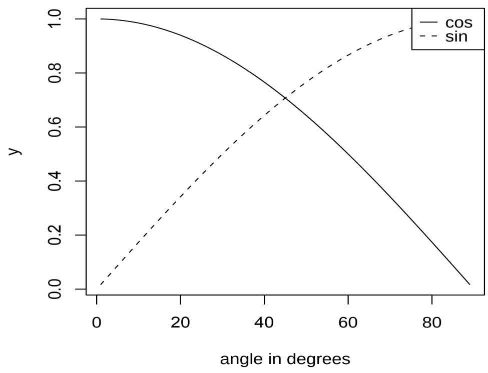


Figure 6.12: $\tan x$

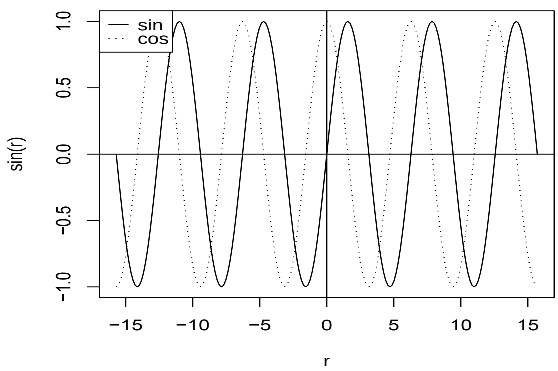


Figure 6.13: Plot of sin and cos